DAG Seminar.

Lecture 1 – 6/9/2021.

1) What is derived algebraic geometry?

Recall Grothendieck's point of view: given a scheme X one has a functor:

$$h_X : (Sch^{aff})^{op} \to Sets.$$
$$S \mapsto Maps(S, X).$$

Moreover, we can understand Sch the cat. of schemes as objects in $PreStk := Fun((Sch^{aff})^{op}, Sets)$ w/ some conditions.

Indeed, $(Sch^{aff})^{op} = CAlg_k$ (I will drop k, a fixed field.) has many topologies (Zariski, étale, flat).

One has $Stk \hookrightarrow PreStk$ the subcat. of $F : CAlg \to Sets$ that are (say) ~~flat~~ étale sheaves.

(One has $L : PreStk \to Stk$ a left adjoint to the inclusion – sheafification functor.)

Def'n: A (q.c.-q.s.) scheme is an object $X \in PreStk$ satisfying: (i) X is an étale sheaf;

(ii) $X \to X \times X$ is representable, i.e. $\forall S \to X,\ S \in Sch^{aff}$ $X \underset{(X \times X)}{\times} S$ is affine.

these words are defined by testing on affines T.

(iii) $\exists\ U \xrightarrow{p} X$ a Zariski cover, i.e.

$U = L\left(\coprod_I S_i\right)$ s.t. $S_i \hookrightarrow X$ open embedding.
affines.

\+ p_i's are jointly surjective.

$Sch \hookrightarrow Stk \hookrightarrow PreStk.$

Rk: This definition agrees w/ locally ringed spaces locally modelled on $Spec(R)$, R a comm. ring. /k.

Derived algebraic geometry generalizes the picture:

$$CAlg \xrightarrow[X]{} Sets$$ in

two ~~one~~ natural (ambitious) ways.

I. $CAlg \rightsquigarrow \mathcal{C}Alg :=$ ∞-category of derived k-algebras.

3 options:

- ∞-cat. of simplicial comm. rings over k;
- ∞-cat. of connective cdga's, i.e. $A^{\cdot} \in Ch(k)^{\leq 0} = Vect^{\leq 0}$ w/ a comm. mult.
- ∞-cat. of $\mathbb{E}_\infty$-algebra (i.e. homotopy coherent comm. mult.) objects in $Vect^{\leq 0}$.

Rk 1: We will talk about these models more later on.

Rk An ∞-category $\mathcal{C}$ (we will spend some time on this) is a

- collection of objects $X \in Ob(\mathcal{C})$;
- for any pair $X, Y \in Ob(\mathcal{C})$ a topological space. $Hom_{\mathcal{C}}(X, Y)$, only meaningful up to homotopy.

II. Sets $\rightsquigarrow$ Spc.

Ex: Spc the ∞-category of topological spaces, where $* \simeq Y$, whenever Y is contractible.
(we will construct Spc precisely once we define ∞-cats)

Sets $\hookrightarrow$ Spc, by $Spc^{\leq k} := \{ X \in Spc \mid \pi_i(X) = 0 \ \forall i > k \}$.

(Notice this makes sense, since $\pi_i(X) = \pi_i(Y)$ whenever $X \sim Y$ homotopy equivalent.) Clearly, $Sets \simeq Spc^{\leq 0}$.
$Grpd \simeq Spc^{\leq 1}$.

One has functors $Spc \to Spc^{\leq k}$ given by killing higher cells.

Similarly, one has $\mathcal{C}Alg^{\geq -n} := \{ A^\bullet \in CAlg \mid H^i(A^\bullet) = 0 \ \forall i > k \}$.

and maps $\mathcal{C}Alg \to \mathcal{C}Alg^{\geq -n}$.

Notice: $\mathcal{C}Alg^{\geq 0} \simeq CAlg$.

$$
\begin{tikzcd}
\mathcal{C}\mathrm{Alg} \arrow[rr, "\mathcal{X}"] & & \mathrm{Spc} \arrow[d] \\
\vdots \arrow[u] & & \vdots \arrow[d] \\
\mathcal{C}\mathrm{Alg}^{\geq -n} \arrow[u] & & \mathrm{Spc}^{\leq k} \arrow[d] \\
\vdots \arrow[u] & & \vdots \arrow[d] \\
\mathcal{C}\mathrm{Alg}^{\geq -1} \arrow[u] & & \mathrm{Spc}^{\leq 1} \simeq \mathrm{Grpd} \arrow[d] \\
\mathcal{C}\mathrm{Alg}^{\geq 0} \simeq \mathrm{CAlg} \arrow[u] \arrow[uuuuurr, "\mathcal{X}_0"] \arrow[urr, "{[X/G]}"] \arrow[rr, "X"'] & & \mathrm{Sets}
\end{tikzcd}
$$

Here: • X is a (classical) scheme.

• $[X/G]$ is an algebraic stack (Artin) ~~(Deligne-Mumford~~

• $\mathcal{X}_0$ is a higher (algebraic) stack (Simpson)

• $\mathcal{X}$ is a derived stack.

Derived schemes: $\underline{\mathrm{Sch}} \hookrightarrow \underline{\mathrm{Stk}} \hookrightarrow \underline{\mathrm{PrStk}}$.

given by natural generalization of conditions: (i), (ii) & (iii)